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Institution: Beijing-Dublin International College
Module: University Physics 2 (BDIC2008J)
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Problem Set 9

Optical Wave Diffraction & Polarization

Problem 1. *The distance between the first and fifth minima of a single-slit diffraction pattern is* 0.35 mm *with the screen* 40 cm *away from the slit, when light of wavelength* 550 nm *is used.* ***(a)*** *Find the slit width.* ***(b)*** *Calculate the angle θ of the first diffraction minimum.*

Solution. **(a)** We use $a\sin\theta = m\lambda$ to calculate the separation between the first ($m_1 = 1$) and fifth ($m_2 = 5$) minima:

$$\Delta x = f\Delta\sin\theta = f\Delta\left(\frac{m\lambda}{a}\right) = \frac{f\lambda}{a}\Delta m = \frac{f\lambda}{a}(m_2 - m_1)$$

Solving for the slit width, we obtain

$$a = \frac{f\lambda(m_2 - m_1)}{\Delta y} = \frac{(400\,\text{mm})(550\times10^{-6}\,\text{mm})(5-1)}{0.35\,\text{mm}} = 2.5\,\text{mm}$$

(b) For $m = 1$,

$$\sin\theta = \frac{m\lambda}{a} = \frac{(1)(550\times10^{-6}\,\text{mm})}{2.5\,\text{mm}} = 2.2\times10^{-4}$$

The angle is $\theta = \arcsin\left(2.2\times10^{-4}\right) = 2.2\times10^{-4}\,\text{rad}$. □

Problem 2. *A plane wave of wavelength* $590\,\mathrm{nm}$ *is incident on a slit with a width of* $a = 0.40\,\mathrm{mm}$. *A thin converging lens of focal length* $+70\,\mathrm{cm}$ *is placed between the slit and a viewing screen and focuses the light on the screen.* ***(a)*** *How far is the screen from the lens?* ***(b)*** *What is the distance on the screen from the centre of the diffraction pattern to the first minimum?*

Solution. **(a)** A plane wave is incident on the lens so it is brought to focus in the focal plane of the lens, a distance of $70\,\mathrm{cm}$ from the lens.

(b) Waves leaving the lens at an angle θ to the forward direction interfere to produce an intensity minimum if $a \sin\theta = m\lambda$, where a is the slit width, λ is the wavelength, and m is an integer. The distance on the screen from the centre of the pattern to the minimum is given by $x = f\tan\theta$, where f is the distance from the lens to the screen. For the conditions of this problem,

$$\sin\theta = \frac{m\lambda}{a} = \frac{(1)(590 \times 10^{-9}\,\mathrm{m})}{0.40 \times 10^{-3}\,\mathrm{m}} = 1.475 \times 10^{-3}$$

This means $\theta = 1.475 \times 10^{-3}\,\mathrm{rad}$ and

$$x = (0.70\,\mathrm{m})\tan\left(1.475 \times 10^{-3}\,\mathrm{rad}\right) = 1.0 \times 10^{-3}\,\mathrm{m}$$

□

Problem 3. *A single slit is illuminated by light of wavelengths λ_a and λ_b, chosen so that the first diffraction minimum of the λ_a component coincides with the second minimum of the λ_b component. **(a)** If $\lambda_b = 350\,\text{nm}$, what is λ_a? For what order number m_b (if any) does a minimum of the λ_b component coincide with the minimum of the λ_a component in the order number **(b)** $m_a = 2$ and **(c)** $m_a = 3$?*

Solution. **(a)** The condition for a minimum in a single-slit diffraction pattern is given by

$$a \sin\theta = m\lambda$$

where a is the slit width, λ is the wavelength, and m is an integer. For $\lambda = \lambda_a$ and $m = 1$, the angle θ is the same as for $\lambda = \lambda_b$ and $m = 2$. Thus,

$$\lambda_a = 2\lambda_b = 2(350\,\text{nm}) = 700\,\text{nm}$$

(b) Let m_a be the integer associated with a minimum in the pattern produced by light with wavelength λ_a, and let m_b be the integer associated with a minimum in the pattern produced by light with wavelength λ_b. A minimum in one pattern coincides with a minimum in the other if they occur at the same angle. This means $m_a\lambda_a = m_b\lambda_b$. Since $\lambda_a = 2\lambda_b$, the minima coincide if $2m_a = m_b$. Consequently, every other minimum of the λ_b pattern coincides with a minimum of the λ_a pattern. With $m_a = 2$, we have $m_b = 4$.

(c) With $m_a = 3$, we have $m_b = 6$. $\square$

Problem 4. *Monochromatic light of wavelength* $441\,\mathrm{nm}$ *is incident on a narrow slit. On a screen* $2.00\,\mathrm{m}$ *away, the distance between the second diffraction minimum and the central maximum is* $1.50\,\mathrm{cm}$. ***(a)*** *Calculate the angle of diffraction* θ *of the second minimum.* ***(b)*** *Find the width of the slit.*

Solution. **(a)** $\theta = \arcsin(1.50\,\mathrm{cm}/2.00\,\mathrm{cm}) = 0.430°$.

(b) For the mth diffraction minimum, $a\sin\theta = m\lambda$. We solve for the slit width:

$$a = \frac{m\lambda}{\sin\theta} = \frac{(2)(441\,\mathrm{nm})}{\sin 0.430°} = 0.118\,\mathrm{mm}$$

□

Problem 5. *A slit* $1.00\,\mathrm{mm}$ *wide is illuminated by light of wave-length* $589\,\mathrm{nm}$. *We see a diffraction pattern on a screen* $3.00\,\mathrm{m}$ *away. What is the distance between the first two diffraction minima on the same side of the central diffraction maximum?*

Solution. The condition for a minimum of intensity in a single-slit diffraction pattern is given by $a\sin\theta = m\lambda$, where a is the slit width, λ is the wavelength, and m is an integer. To find the angular position of the first minimum to one side of the central maximum, we set $m = 1$,

$$\theta_1 = \arcsin\left(\frac{\lambda}{a}\right) = \arcsin\left(\frac{589\times10^{-9}\,\mathrm{m}}{1.00\times10^{-3}\,\mathrm{m}}\right) = 5.89\times10^{-4}\,\mathrm{rad}$$

If f is the distance from the slit to the screen, the distance on the screen from the centre of the pattern to the minimum is

$$x_1 = f\tan\theta_1 = (3.00\,\mathrm{m})\tan(5.89\times10^{-4}\,\mathrm{rad}) = 1.767\times10^{-3}\,\mathrm{m}$$

To find the second minimum, we set $m = 2$,

$$\theta_2 = \arcsin\left(\frac{2(589\times10^{-9}\,\mathrm{m})}{1.00\times10^{-3}\,\mathrm{m}}\right) = 1.178\times10^{-3}\,\mathrm{rad}$$

The distance from the center of the pattern to this second minimum is

$$x_2 = f\tan\theta_2 = (3.00\,\mathrm{m})\tan(1.178\times10^{-3}\,\mathrm{rad}) = 3.534\times10^{-3}\,\mathrm{m}$$

The separation of the two minima is

$$\Delta x = x_2 - x_1 = 3.534\,\mathrm{mm} - 1.767\,\mathrm{mm} = 1.77\,\mathrm{mm}$$

DISCUSSION The angles θ_1 and θ_2 found above are quite small. In the small-angle approximation, $\sin\theta \approx \tan\theta \approx \theta$, and the separation between two adjacent diffraction minima can be approximated as

$$\Delta x = f(\tan\theta_{m+1} - \tan\theta_m) \approx f(\theta_{m+1} - \theta_m) = \frac{f\lambda}{a}$$

□

Problem 6. *Light of wavelength* 600 nm *is incident normally on a diffraction grating. Two adjacent maxima occur at angles given by* $\sin\theta = 0.2$ *and* $\sin\theta = 0.3$. *The fourth-order maxima are missing.* ***(a)*** *What is the separation between adjacent slits?* ***(b)*** *What is the smallest slit width this grating can have? For that slit width, what are the* ***(c)*** *largest,* ***(d)*** *second largest, and* ***(e)*** *third largest values of the order number* m *of the maxima produced by the grating?*

Solution. Maxima of a diffraction grating pattern occur at angles θ given by $d\sin\theta = m\lambda$, where d is the slit separation, λ is the wavelength, and m is an integer. If two lines are adjacent, then their order numbers differ by unity. Let m be the order number for the line with $\sin\theta = 0.2$, and $m+1$ be the order number for the line with $\sin\theta = 0.3$. Then,

$$0.2d = m\lambda, \qquad 0.3d = (m+1)\lambda$$

(a) We subtract the first equation from the second to obtain $0.1d = \lambda$, or

$$d = \lambda/0.1 = (600\times10^{-9}\,\text{m})/0.1 = 6.0\times10^{-6}\,\text{m}$$

(b) Minima of the single-slit diffraction pattern occur at angles θ given by $a\sin\theta = m\lambda$, where a is the slit width. Since the fourth-order interference maximum is missing, it must fall at one of these angles. If a is the smallest slit width for which this order is missing, the angle must be given by $a\sin\theta = \lambda$. It is also given by $d\sin\theta = 4\lambda$, so

$$a = d/4 = (6.0\times10^{-6}\,\text{m})/4 = 1.5\times10^{-6}\,\text{m}$$

(c) First, we set $\theta = 90^\circ$ and find the largest value of m for which $m\lambda < d\sin\theta$. This is the highest order that is diffracted toward the screen. The condition is the same as $m < d/\lambda$ and since

$$d/\lambda = (6.0\times10^{-6}\,\text{m})/(600\times10^{-9}\,\text{m}) = 10$$

the highest order seen is the $m = 9$ order. The fourth and eighth orders are missing, so the observable orders are $m = 0, 1, 2, 3, 5, 6, 7$, and 9. Thus, the largest value of the order number is $m = 9$.

(d) Using the result obtained in **(c)**, the second largest value of the order number is $m = 7$.

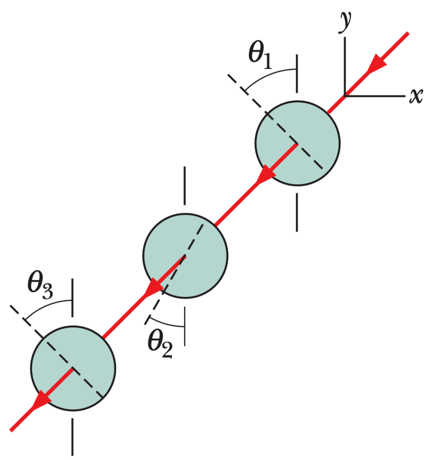
(e) Similarly, the third largest value of the order number is $\text{m} = 6$.

DISCUSSION Interference maxima occur when $d\sin\theta = m\lambda$, while the condition for diffraction minima is $a\sin\theta = m'\lambda$. Thus, a particular interference maximum with order m may coincide with the diffraction minimum of order m'. The value of m is given by

$$\frac{d\sin\theta}{a\sin\theta} = \frac{m\lambda}{m'\lambda} \quad\Rightarrow\quad m = \left(\frac{d}{a}\right)m'$$

Since $m = 4$ when $m' = 1$, we conclude that $d/a = 4$. Thus, $m = 8$ would correspond to the second diffraction minimum (m'). $\square$

Problem 7. *In the figure below, initially unpolarized light is sent into a system of three polarizing sheets whose polarizing directions make angles of $\theta_1 = 40°, \theta_2 = 20°$, and $\theta_3 = 40°$ with the direction of the y axis. What percentage of the light's initial intensity is transmitted by the system? (Hint: Be careful with the angles.)*



Solution. Unpolarized light becomes polarized when it is sent through a polarizing sheet. In this problem, three polarizing sheets are involved, we work through the system sheet by sheet, applying either the one-half rule or the cosine-squared rule. Let I_0 be the intensity of the unpolarized light that is incident on the first polarizing sheet. The transmitted intensity is, by one-half rule, $I_1 = \frac{1}{2}I_0$, and the direction of polarization of the transmitted light is $\theta_1 = 40°$ *counterclockwise* from the y axis in the diagram. For the second sheet (and the third one as well), we apply the cosine-squared rule:

$$I_2 = I_1 \cos^2 \theta_2'$$

where θ_2' is the angle between the direction of polarization that is incident on that sheet and the polarizing direction of the sheet. The polarizing direction of the second sheet is $\theta_2 = 20°$ *clockwise* from the y axis, so $\theta_2' = 40° + 20° = 60°$. The transmitted intensity is

$$I_2 = I_1 \cos^2 60° = \frac{1}{2} I_0 \cos^2 60°$$

and the direction of polarization of the transmitted light is $20°$ clockwise from the y axis. The polarizing direction of the third sheet is $\theta_3 = 40°$ *counterclockwise* from the y axis. Consequently, the angle between the direction of polarization of the light incident on that sheet and polarizing direction of the sheet is $40° + 20° = 60°$. The transmitted intensity is

$$I_3 = I_2 \cos^2 60° = \frac{1}{2} I_0 \cos^4 60° = 3.1 \times 10^{-2} I_0$$

Thus, 3.1% of the light's initial intensity is transmitted.

DISCUSSION When two polarizing sheets are crossed ($\theta = 90°$), no light passes through and the transmitted intensity is zero. □

Problem 8. *We want to rotate the direction of polarization of a beam of polarized light through* $90°$ *by sending the beam through one or more polarizing sheets.* ***(a)*** *What is the minimum number of sheets required?* ***(b)*** *What is the minimum number of sheets required if the transmitted intensity is to be more than 60% of the original intensity?*

Solution. A polarizing sheet can change the direction of polarization of the incident beam since it allows only the component that is parallel to its polarization direction to pass. The $90°$ rotation of the polarization direction cannot be done with a single sheet. If a sheet is placed with its polarizing direction at an angle of $90°$ to the direction of polarization of the incident radiation, no radiation is transmitted.

(a) The $90°$ rotation of the polarization direction can be done with two sheets. We place the first sheet with its polarizing direction at some angle θ, between 0 and $90°$, to the direction of polarization of the incident radiation. Place the second sheet with its polarizing direction at $90°$ to the polarization direction of the incident radiation. The transmitted radiation is then polarized at $90°$ to the incident polarization direction. The intensity is

$$I = I_0 \cos^2\theta \cos^2(90° - \theta) = I_0 \cos^2\theta \sin^2\theta$$

where I_0 is the incident radiation. If θ is not 0 or $90°$, the transmitted intensity is not zero.

(b) Consider n sheets, with the polarizing direction of the first sheet making an angle of $\theta = 90°/n$ relative to the direction of polarization of the incident radiation. The polarizing direction of each successive sheet is rotated $90°/n$ in the same sense from the polarizing direction of the previous sheet. The transmitted radiation is polarized, with its direction of polarization making an angle of $90°$ with the direction of polarization of the incident radiation. The intensity is

$$I = I_0 \cos^{2n}(90°/n)$$

We want the smallest integer value of n for which this is greater than $0.60I_0$. We start with $n = 2$ and calculate $\cos^{2n}(90°/n)$. If the result is greater than 0.60, we have obtained the solution. If it is less, increase n by 1 and try again. We repeat this process, increasing n by 1 each time, until we have a value for which $\cos^{2n}(90°/n)$ is greater than 0.60. The first one will be $n = 5$.

DISCUSSION The intensities associated with $n = 1$ to 5 are

$$\begin{aligned}
I_{n=1} &= I_0 \cos^2(90°) = 0 \\
I_{n=2} &= I_0 \cos^4(45°) = I_0/4 = 0.25I_0 \\
I_{n=3} &= I_0 \cos^6(30°) = 0.422I_0 \\
I_{n=4} &= I_0 \cos^8(22.5°) = 0.531I_0 \\
I_{n=5} &= I_0 \cos^{10}(18°) = 0.605I_0
\end{aligned}$$

□

Problem 9. *Light that is traveling in water (with an index of refraction of 1.33) is incident on a plate of glass (with index of refraction 1.53). At what angle of incidence does the reflected light end up fully polarized?*

Solution. A reflected wave will be fully polarized if it strikes the boundary at the Brewster angle. The angle of incidence for which reflected light is fully polarized is given by

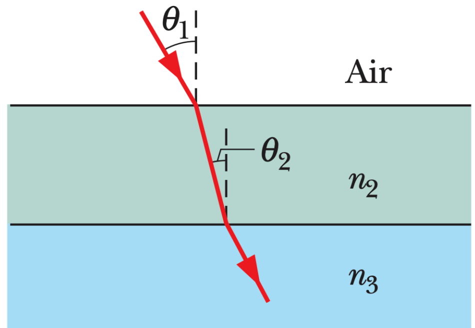
$$\theta_B = \arctan\left(\frac{n_2}{n_1}\right)$$

where n_1 is the index of refraction for the medium of incidence, and n_2 is the index of refraction for the second medium. The angle θ_B is called the Brewster angle. With $n_1 = 1.33$ and $n_2 = 1.53$, we obtain

$$\theta_B = \arctan\left(\frac{n_2}{n_1}\right) = \arctan\left(\frac{1.53}{1.33}\right) = 49.0°$$

In general, reflected light is partially polarized, having components both parallel and perpendicular to the plane of incidence. However, it can be completely polarized when incident at the Brewster angle. □

Problem 10. *In the figure below, a light ray in air is incident on a flat layer of material 2 that has an index of refraction* $n_2 = 1.5$*. Beneath material 2 is material 3 with an index of refraction* n_3*. The ray is incident on the air – material 2 interface at the Brewster angle for that interface. The ray of light refracted into material 3 happens to be incident on the material 2 – material 3 interface at the Brewster angle for that interface. What is the value of* n_3*?*



Solution. Since the layers are parallel, the angle of refraction regarding the first surface is the same as the angle of incidence regarding the second surface. We recall the refracted angle is the complement of the incident angle

$$\theta_2 = (\theta_1)_c = 90^\circ - \theta_1$$

We apply Brewster's angle to both refractions, setting up a product

$$\left(\frac{n_2}{n_1}\right)\left(\frac{n_3}{n_2}\right) = (\tan\theta_{B1\to 2})(\tan\theta_{B2\to 3}) \qquad \Rightarrow \qquad \frac{n_3}{n_1} = (\tan\theta_1)(\tan\theta_2)$$

Now, since θ_2 is the complement of θ_1 we have

$$\tan\theta_2 = \tan(\theta_1)_c = \frac{1}{\tan\theta_1}$$

Therefore, the product of tangents cancel and we obtain $n_3/n_1 = 1$. Consequently, the third medium is air: $n_3 = 1.0$. □